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Simultaneous Approximation of a Uniformly Bounded Set of Real Valued Functions

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Let F be a set of uniformly bounded, real valued functions on $[a, b]$ and S a non-empty family of real valued functions on $[a, b]$. If there exists an $s^* \in S$ such that

$$\inf_{s \in S} \sup_{f \in F} \|f - s\| = \sup_{f \in F} \|f - s^*\|,$$

then s^* is called a best simultaneous approximation to F. This definition was given by Diaz and McLaughlin [1]. They proved that best simultaneous approximation of F is equivalent to best simultaneous approximation of the two functions F^+ and F^- where

$$F^+ = \inf_{\delta > 0} \sup_{0 \leqslant |x-y| < \delta} \sup_{f \in F} f(y)$$

and

$$F^- = \sup_{\delta > 0} \inf_{0 \leqslant |x-y| < \delta} \inf_{f \in F} f(y).$$

In this note, using a different approach, we show that best simultaneous approximation to F is equivalent to best simultaneous approximation of the two functions $\sup_{f \in F} f$ and $\inf_{f \in F} f$. The same result is also obtained for simultaneous approximation of F in the L_1 norm.

THEOREM 1. *If $s^* \in S$ is a best approximation to F then*

$$\sup_{f \in F} \|f - s^*\| = \| \, |(\sup_{f \in F} f + \inf_{f \in F} f)/2 - s^*| + (\sup_{f \in F} f - \inf_{f \in F} f)/2 \|. \tag{1}$$

We first prove the following Lemma.

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LEMMA 1. *Let A be a bounded set of real numbers and r be any real number. Then*

$$\sup_{a \in A} |a - r| = |(\alpha + \beta)/2 - r| + (\alpha - \beta)/2 \tag{2}$$

where $\alpha = \sup_{a \in A} a$ *and* $\beta = \inf_{a \in A} a$.

Proof. It is sufficient to prove that

$$\sup_{a \in A} |a| = |(\alpha + \beta)/2| + (\alpha - \beta)/2. \tag{3}$$

In fact, (2) follows by considering $A - r$ instead of A.

Case 1. If $\alpha + \beta > 0$ then $\alpha > -\beta = -\inf_{a \in A} a = \sup_{a \in A} (-a)$. Hence $\sup_{a \in A} |a| = \sup_{a \in A} a = \alpha$. It is clear that in this case the right-hand side of (3) also equals α.

Case 2. If $\alpha + \beta < 0$ then $\alpha < -\beta = -\inf_{a \in A} a = \sup_{a \in A} (-a)$. Hence $\sup_{a \in A} |a| = \sup_{a \in A} (-a) = -\beta$. In this case the right side of (3) alsoequals $-\beta$. Hence (3) holds.

Proof of Theorem 1. Let $s \in S$ and $x \in [a, b]$. Then by Lemma 1

$$\begin{aligned} \sup_{f \in F} |f(x) - s(x)| = {} & |\{\sup_{f \in F} f(x) + \inf_{f \in F} f(x)\}/2 - s(x)| \\ & + \sup_{f \in F} f(x) - \inf_{f \in F} f(x)\}/2. \end{aligned} \tag{4}$$

Now, we take the supremum of both sides of (4) over $[a, b]$:

$$\sup_{f \in F} \|f - s\| = \| |(\sup_{f \in F} f + \inf_{f \in F} f)/2 - s| + (\sup_{f \in F} f - \inf_{f \in F} f)/2\|.$$

Then by taking the infimum over S we obtain (1). This completes the proof.

If f_1 and f_2 are any two real valued functions on $[a, b]$ then $\forall x \in [a, b]$ $\sup_{i=1,2} f_i(x) + \inf_{i=1,2} f_i(x) = f_1(x) + f_2(x)$ and $\sup_{i=1,2} f_i(x) - \inf_{i=1,2} f_i(x) = |\{f_1(x) - f_2(x)\}/2|$. Therefore, we obtain the following result, which had been proven in [2], as a special case of Theorem 1:

$$\begin{aligned} & \inf_{s \in S} \max\{\|f_1 - s\|, \|f_2 - s\|\} \\ & \quad = \inf_{s \in S} \| |(f_1 + f_2)/2 - s| + (f_1 - f_2)/2\|. \end{aligned} \tag{5}$$

THEOREM 2. *If s^* is a best simultaneous approximation to F, then*

$$\sup_{f \in F} \|f - s^*\| = \max\{\|\sup_{f \in F} f - s^*\|, \|\inf_{f \in F} f - s^*\|\}. \tag{6}$$

That is, s^* is a best simultaneous approximation to F if and only if it is a best simultaneous approximation to $\sup_{f \in F} f$ and $\inf_{f \in F} f$.

Proof. Substituting $f_1 = \sup_{f \in F} f$ and $f_2 = \inf_{f \in F} f$ in (5) we get

$$\max\{\|\sup_{f \in F} f - s^*\|, \|\inf_{f \in F} f - s^*\|\}$$
$$= \| |(\sup_{f \in F} f + \inf_{f \in F} f)/2 - s^*| + (\sup_{f \in F} f - \inf_{f \in F} f)/2\|.$$

Together with (1) we obtain (6).

Next we consider simultaneous approximation problem in L_1 norm. Now, let F be a set of uniformly bounded integrable functions on $[a, b]$ and S be a non-empty set of integrable functions on $[a, b]$. Then $s^* \in S$ is said to be *a best simultaneous approximation to F in L_1 norm* if

$$\inf_{s \in S} \int_a^b \sup_{f \in F} |f(x) - s(x)| \, dx = \int_a^b \sup_{f \in F} |f(x) - s^*(x)| \, dx.$$

By integrating (4) we obtain the following theorem.

THEOREM 3. *If $s^* \in S$ is a best simultaneous approximation to F in L_1 norm then*

$$\int_a^b \sup_{f \in F} |f(x) - s^*(x)| \, dx$$
$$= \int_a^b |(\sup_{f \in F} f + \inf_{f \in F} f)/2 - s^*(x)| \, dx + \int_a^b (\sup_{f \in F} f - \inf_{f \in F} f)/2 \, dx. \tag{7}$$

When we consider only two functions f_1 and f_2 it is shown in [3] that s^* is a best simultaneous approximation to f_1 and f_2 in L_1 norm if

$$\int_a^b \max\{|f_1(x) - s^*(x)|, |f_2(x) - s^*(x)|\} \, dx$$
$$= \int_a^b |(f_1 + f_2)/2 - s^*| \, dx + \int_a^b |(f_1 - f_2)/2| \, dx. \tag{8}$$

Here, we see that this result is a special case of Theorem 3.

THEOREM 4. *s^* is a best simultaneous approximation to F in L_1 norm if and only if*

$$\int_a^b \sup_{f \in F} |f(x) - s^*(x)| \, dx$$

$$= \int_a^b \max\{|\sup_{f \in F} f(x) - s^*(x)|, |\inf_{f \in F} f(x) - s^*(x)|\} \, dx. \tag{9}$$

That is, simultaneous approximation to F in L_1 norm is equivalent to the simultaneous approximation of the two functions $\sup_{f \in F} f$ and $\inf_{f \in F} f$ in L_1 norm.

Proof. We substitute $f_1 = \sup_{f \in F} f$ and $f_2 = \inf_{f \in F} f$ in (8). Then together with (7) we obtain (9).

REFERENCES

1. J. B. DIAZ AND H. W. MCLAUGHLIN, Simultaneous approximation of a set of bounded real functions, *Math. Comp.* **23** (1969), 583–594.
2. J. B. DIAZ AND H. W. MCLAUGHLIN, On simultaneous Chebyshev approximation and Chebyshev approximation with additive weight, *J. Approx. Theory* **6** (1972), 68–71.
3. G. M. PHILLIPS AND B. N. SAHNEY, On best simultaneous approximation in the L_1 and L_2 norms, *in* "Theory of Approximation with Applications" (Law and Sahney, Eds.), pp. 213–219, Academic Press, New York, 1976.